Homework #1 of Topology II

Due Date: Jan 31, 2018

1. Determine the smooth structure of $n - sphere$

$$S^n = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1} = 1\}.$$

   (Find out the local coordinates and the transition functions. Do not use Regular Value Theorem.)

2. Let V be a $n - dimensional$ linear subspace of $\mathbb{R}^N$. Prove that V is a smooth manifold and $T_v V = V$ for all $v \in V$.

3. (a) Show that for any manifolds X and Y,

$$T_{(x,y)} X \times Y = T_x X \times T_y Y.$$

   (b) Let $f : X \times Y \to X$ be the projection map $(x, y) \to x$. Show that the differential map

$$f_* : T_x X \times T_y Y \to T_x X$$

   is the projection $(v, w) \to v$.

   (c) Let $f : X \to X'$ and $g : Y \to Y'$ be any smooth maps. Prove that

$$(f \times g)_* = f_* \times g_*.$$

4. (a) Let $M(n)$ be the space of all $n \times n$ matrices with real entries. Find a smooth structure on $M(n)$ such that it is $n^2 - dimensional$ manifold.

   (b) Let $S(n) \subset M(n)$ be the submanifold consisting of all symmetric matrices. Show the map

   $f : M(n) \to S(n)$ via $f(A) = AA^t$ is a smooth map, where A^t is the transpose of A.

   (c) Find the differential $f_* : T_A M(n) \to T_{f(A)} S(n)$.(Hint: you can use the formula $\lim_{t \to 0} \dfrac{f(A + tB) - f(A)}{t}$.)

   (d) Determine if the identity matrix $I \in S(n)$ is the regular value of f and find the dimension of the orthogonal group $O(n) = \{A \in M(n) | AA^t = I\}$.

5. Verify the tangent space to $O(n)$ at the identity matrix I is the vector space of skew symmetric $n \times n$ matrices, i.e. matrices A satisfying $A^t = -A$.

6. (a) If X is a compact manifold and Y is a connected manifold, show that every submersion $f : X \to Y$ is surjective. (b) Show that there exist no submersions of compact manifolds into Euclidean spaces.

7. (a) Suppose that $A : \mathbb{R}^k \to \mathbb{R}^n$ is a linear map and V is a vector subspace of $\mathbb{R}^n$. Check that $A \pitchfork V$ implies that $A(\mathbb{R}^k) + V = \mathbb{R}^n$.

   (b) If V and W are linear subspaces of $\mathbb{R}^n$, show that $V \pitchfork W$ implies $V + W = \mathbb{R}^n$.

8. Let X and Z be transversal submanifolds of Y. Prove that if $y \in X \cap Z$, then
$$T_y(X \cap Z) = T_yX \cap T_yZ.$$

9. Let $f : X \to Y$ be a smooth map transversal to a submanifold Z in Y. Then $W = f^{-1}(Z)$ is a submanifold of X. Show that T_xW is the preimage of $T_{f(x)}(Z)$ under the linear map $f_* : T_xX \to T_{f(x)}Y$.